An example Rocq blueprint project

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Chapter 1

The naturals

In this chapter we give some basic definitions and lemmas about the natural numbers.

Definition 1. The set of naturals, denoted $\mathbb{N}$, is the set of numbers $\{0, 1, 2, ...\}$.

Lemma 2. *Let n be a natural number. Then $n = n + 0$.*

Proof. $\square$

Lemma 3. *Let n and m be natural numbers. Then $(n + m) + 1 = n + (m + 1)$.*

Proof. $\square$

Lemma 4. *Addition is commutative, i.e. $n + m = m + n$ for all $m, n \in \mathbb{N}$.*

Proof. $\square$

Lemma 5. *Let n be a natural number. Then $2 * n = n + 2$.*

Proof. The proof is trivial and left as an exercise to the reader. $\square$

Chapter 2

The even numbers

In this chapter we introduce two different notions of evenness of a natural number. We also provide an algorithmic criterion, and prove all three notions are equivalent.

2.1 An induction criterion with a step size of two

Before diving into evenness, we first give a very useful induction principle that allows us to advance by two in Lemma 6.

Lemma 6. *Let P be a proposition about natural numbers such that $P(0)$ and $P(1)$ hold and for all $n \in \mathbb{N}$, $P(n)$ implies that $P(n+2)$ holds. Then $P(n)$ holds for all $n \in \mathbb{N}$.*

Proof. Instead of proving that $P(n)$ holds for all $n \in \mathbb{N}$, we prove the stronger statement that both $P(n)$ and $P(n+1)$ holds for all $n \in \mathbb{N}$. To do so, we proceed by induction on n. Indeed, in the base case, if $n = 0$, then $P(0)$ and $P(1)$ hold by assumption.

Now assume that $P(n)$ and $P(n+1)$ both hold for some n. We now prove that $P(n+1)$ and $P(n+2)$ hold. It follows from the inductive hypothesis that $P(n+1)$ holds. Furthermore, since $P(n)$ holds, it follows from the assumptions of the lemma that $P(n+2)$ also holds. Hence $P(n+1)$ and $P(n+2)$ both hold, completing the induction and therefore the lemma. □

2.2 Different definitions of evenness

We begin by laying out three definitions of evenness, a multiplicative definition in Definition 7, an additive definition in Definition 8 and finally an algorithmic criterion in Definition 9.

Definition 7. Let $n \in \mathbb{N}$ be a natural number. We say that m is *multiplicatively even* if there exists an $m \in \mathbb{N}$ such that $n = 2m$.

Definition 8. The set of *additively even* numbers is the smallest set of naturals E such that

(a) $0 \in E$ and

(b) if $n \in E$ then $n + 2 \in E$ as well.

Definition 9. Let $\mathbb{B} = \{\texttt{true}, \texttt{false}\}$. Define the function $\texttt{iseven} : \mathbb{N} \to \mathbb{B}$ by

$$\texttt{iseven}(n) = \begin{cases} \texttt{true} & \text{if } n = 0, \\ \texttt{not iseven}(n-1) & \text{otherwise,} \end{cases}$$

where $\texttt{not true} = \texttt{false}$ and $\texttt{not false} = \texttt{true}$.

The remainder of this section will be dedicated to proving Theorem 10.

Theorem 10. *Let n be a natural number. Then the following are equivalent:*

(i) n is multiplicatively even, as defined in Definition 7,

(ii) n is additively even, as defined in Definition 8,

(iii) $\texttt{iseven}(n) = \texttt{true}$, where $\texttt{iseven}$ is the function defined in Definition 9.

In order to prove Theorem 10, we start by proving two lemmas stating that the various notions of evenness are preserved under addition by 2.

Lemma 11. *Let n be a multiplicatively even number. Then $n + 2$ is also multiplicatively even.*

Proof. Since n is multiplicatively even, there exists $m \in \mathbb{N}$ such that $n = 2m$. Then $n + 2 = 2m + 2 = 2(m + 1)$, hence $n + 2$ is also multiplicately even, as required. □

Lemma 12. *Let n be a natural number. Then n is additively even is and only if $n + 2$ is additively even.*

Proof. ($\Rightarrow$) Assume n is additively even. Then $n + 2$ is also additively even, since the set of additively even numbers is closed under adding 2.

($\Leftarrow$) Now assume $n + 2$ is additively even. If n was not additively even, then $E \setminus \{n + 2\}$ would still be a set containing 0 that is closed under adding 2. But the set of additively even numbers E was defined to be the least such set. Therefore n must be additively even. □

Using these lemmas it is straightforward to prove the following two equivalences.

Lemma 13. *Let n be a natural number. Then n is additively even is and only if $\texttt{iseven}(n) = \texttt{true}$.*

Proof. We proceed by induction with step size two on n, from Lemma 6. For our base cases, when $n = 0$ we have that 0 is additively even and $\texttt{iseven}(0) = \texttt{true}$ by definition. When $n = 1$, 1 is not additively even, since if it were, there would need to exist some $m \in E$ such that $m+2 = 1$, but this is not possible for naturals. Furthermore, $\texttt{iseven}(1) = \texttt{not iseven}(0) = \texttt{false} \neq \texttt{true}$, so the statement holds for $n = 1$ as well.

Now assume the equivalence holds for n. We prove it also holds for $n + 2$. To this end, note that $n+2$ is additively even if and only if n is additively even by Lemma 12. Furthermore, $\texttt{iseven}(n+2) = \texttt{not iseven}(n+1) = \texttt{not not iseven}(n) = \texttt{iseven}(n)$. Therefore the statement for $n+2$ is equivalent to the statement for n and hence holds. This completes the induction. □

Lemma 14. *Let n be a natural number. Then n is multiplicatively even is and only if n is additively even.*

Proof. ($\Rightarrow$) Assume that n is multiplicatively even. Then there exists $m \in \mathbb{N}$ such that $n = 2m$. By Lemma 13, to prove that n is additively even, it suffice to prove that $\texttt{iseven}(2m) = \texttt{true}$. We proceed by induction on m. When $m = 0$, clearly $\texttt{iseven}(2m) = \texttt{iseven}(0) = \texttt{true}$.

Now assume that $\texttt{iseven}(2m) = \texttt{true}$. We want to prove that $\texttt{iseven}(2(m + 1)) = \texttt{true}$ as well. But then

$$\texttt{iseven}(2(m+1)) = \texttt{iseven}(2m+2) = \texttt{not iseven}(2m+1) = \texttt{not not iseven}(2m) = \texttt{iseven}(2m) = \texttt{true}.$$

This completes the induction.

($\Leftarrow$) Assume that n is additively even. We proceed by induction with step size two on n, from Lemma 6. For our base cases, when $n = 0$, n is additively even and $n = 2 \cdot 0$ is also multiplicatively even. When $n = 1$, then n is not additively even by the same reasoning as in Lemma 13, hence the statement also holds.

Now assume that the statement holds for n. We prove it holds for $n + 2$. To see this note that by Lemma 12, $n+2$ is additively even if and only if n is. If n was not additively even, then we are done. Otherwise, by the inductive hypothesis, it follows that n is multiplicatively even. But then, by Lemma 11, $n + 2$ is also multiplicatively even. This completes the proof. $\square$

We are finally ready to prove Theorem 10.

Proof of Theorem 10. By Lemma 13, conditions (ii) and (iii) are equivalent. By Lemma 14, conditions (i) and (ii) are equivalent. Hence we are done by transitivity of equivalence. $\square$

2.3 Some further results

Definition 15. A natural number p is *prime* if $p \geq 2$ and for every pair of natural numbers $x, y \in \mathbb{N}$, if $p = xy$ then either $x = 1$ or $y = 1$.

Theorem 16 (Goldbach's conjecture)**.** *Let n be an even number greater than 2. Then there exist primes p and q such that $n = p + q$.*