Maximal One-sided Congruences of Full Transformation Monoids

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Joint work with

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Objects of study

Semigroups

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Groups

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- Subsemigroups

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- Subgroups

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- Subsemigroups
- Two-sided congruences

$$(x, y) \in \rho \Rightarrow (sxt, syt) \in \rho$$

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- Normal subgroups

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Groups

- Subgroups
  - subobjects
  - right action on cosets

$$Hx \cdot y = H(xy)$$

  - left action on cosets
- Normal subgroups

Objects of study

Semigroups

- Subsemigroups
- Right congruences

$$(x, y) \in \rho \Rightarrow (xs, ys) \in \rho$$

- Left congruences
- Two-sided congruences

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- Normal subgroups

Normal subgroups of $\mathcal{S}_n$

Theorem

The normal subgroup lattices of $\mathcal{S}_n$ for $n \in \mathbb{N}$ are precisely as follows:

$$
\begin{array}{cccccc}
\mathcal{S}_1 & \mathcal{S}_2 & \mathcal{S}_3 & \mathcal{S}_4 & \mathcal{S}_n & n \geq 5 \\
 & | & | & | & | & \\
 & \{1\} & A_3 & A_4 & A_n & \\
 & & | & | & | & \\
 & & \{1\} & K_4 & \{1\} & \\
 & & & | & & \\
 & & & \{1\} & &
\end{array}
$$

Two-sided congruences of $\mathcal{T}_n$

Theorem (Mal'cev 1952)

Let $n \geq 2$. Then there is a bijection between non-total two-sided congruences of $\mathcal{T}_n$ and normal subgroups of $\mathcal{S}_k$ for all $k \in \{1, \ldots, n\}$.

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Let $n \geq 2$. Then there is a bijection between non-total two-sided congruences of $\mathcal{T}_n$ and normal subgroups of $\mathcal{S}_k$ for all $k \in \{1, \ldots, n\}$.
In particular the two-sided congruence lattice of $\mathcal{T}_n$ is a chain of height $3n-1$ when $n \geq 4$.

$$
\begin{array}{c}
\nabla_{\mathcal{T}_n} \\
| \\
\rho_{\mathcal{S}_n} \\
| \\
\rho_{A_n} \\
| \\
\rho_{I_n} \\
| \\
\rho_{\mathcal{S}_{n-1}} \\
\vdots \\
\rho_{\mathcal{S}_2} \\
| \\
\rho_{I_2} \\
| \\
\rho_{\mathcal{S}_1}
\end{array}
$$

Similar ideas used to describe the two-sided congruence lattices of:

- Other transformation monoids
  - Partial transformation monoid (Šutov 1988),
  - Symmetric inverse monoid (Scheiblich 1973),
- Diagram monoids
  - The full partition monoid, Brauer monoid, Jones monoid etc. (East, Mitchell, Ruškuc, and Torpey 2018),
  - Twisted partition monoids (East and Ruškuc 2022),
- Endomorphism monoids of ordered sets
  - The monoid of all injective order preserving partial transformations on a finite chain (Fernandes 2001),
  - monoids of order-preserving or order-reversing transformations on a finite chain (Fernandes, Gomes, and Jesus 2005).

Right congruences of $\mathcal{T}_n$

Question

Does there exist a description of the right congruence lattice of $\mathcal{T}_n$ à la Mal'cev?

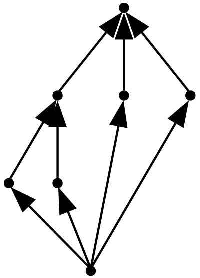
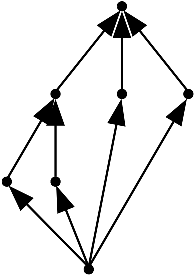


Figure: Right cong. lattice of $\mathcal{T}_2$



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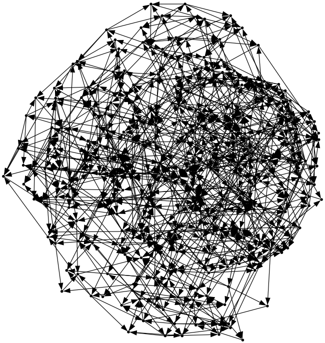

Figure: Right cong. lattice of $\mathcal{T}_3$

n	$\lvert\mathcal{T}_n\rvert$	two-sided	right	left
1	1	1	1	1
2	4	4	7	4
3	27	7	287	180
4	256	11	22'069'828	120'121
5	3125	14	?	?
n	n^n	$3n-1, n \geq 4$	?	?

Table: Number of congruences of $\mathcal{T}_n$

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Brookes, East, Miller, Mitchell, and Ruškuc 2024 give a formula for the height of the right and left congruence lattices of $\mathcal{T}_n$.

Maximal right congruences of $\mathcal{T}_n$

A right congruence ρ of a monoid M is *maximal* if

1. $\rho \neq \nabla_M$ and
2. $\rho \subseteq \sigma \subseteq \nabla_M \Rightarrow \sigma = \rho$ or $\sigma = \nabla_M$.

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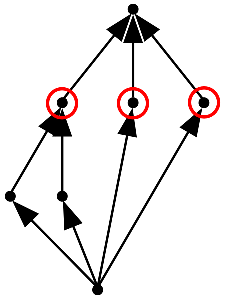

Computational results

n	$\lvert\mathcal{T}_n\rvert$	right	maximal right	left	maximal left
1	1	1	0	1	0
2	4	7	3	4	1
3	27	287	7	180	4
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5	3125	?	≥ 31	?	≥ 51
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n	n^n	?	$2^n - 1$?	?	

Table: Number of congruences of $\mathcal{T}_n$

M. Anagnostopoulou-Merkouri, R. C., J. D. Mitchell, and M. Tsalakou (2024). *Computing finite index congruences of finitely presented semigroups and monoids*. arXiv: 2302.06295 [math.RA]. URL: https://arxiv.org/abs/2302.06295

Code for computing maximal congruences implemented in

libsemigroups_pybind11

The right syntactic congruence

Let S be a semigroup and let $Z \subseteq S$. Define the *right syntactic congruence* induced by Z to be

$$\sim_Z = \{\, (x, y) \in S \times S \mid \forall s \in S,\ xs \in Z \Leftrightarrow ys \in Z \,\}$$

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Lemma

Let ρ be a right congruence on M, let $z \in S$ and let

$$Z = z/\rho = \{ w \in S \mid (z, w) \in \rho \}$$

. Then $\rho \subseteq \sim_Z$.

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Corollary

If S is a monoid and ρ is maximal, then $\rho = \sim_Z$ for some $Z = z/\rho$.

Maximal right congruences of monoids

Let M be a monoid and let ρ be a right congruence on M. Then $1/\rho$ is a submonoid of M. Furthermore $1/\rho$ is a *left division closed* (LDC) submonoid of M, that is for all $x, y \in M$:

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Lemma

Every LDC submonoid N of M arises as $1/\rho$ for some right congruence ρ.

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Lemma

Every LDC submonoid N of M arises as $1/\rho$ for some right congruence ρ.

Lemma

If N is a maximal LDC submonoid of M, then $\sim_N$ is a maximal right congruence.

LDC submonoids of $\mathcal{T}_\Omega$

Let Ω be a set. We write $\mathcal{T}_\Omega = \{f : \Omega \to \Omega \text{ a function}\}$.
Let $\Sigma \subseteq \Omega$ be non-empty. The *setwise stabilizer* of Σ in $\mathcal{T}_\Omega$ is the submonoid

$$\mathrm{Stab}(\Sigma) = \{\, f \in \mathcal{T}_\Omega \mid (\Sigma)f = \Sigma \,\} .$$

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Lemma

$\mathrm{Stab}(\Sigma)$ is an LDC submonoid. When $|\Sigma| < \infty$, $\mathrm{Stab}(\Sigma)$ is also a maximal LDC submonoid.

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Theorem (C., Mitchell, and Péresse 2025+)

Let Ω be a finite set, then every maximal right congruence of $\mathcal{T}_\Omega$ is of the form $\sim_{\operatorname{Stab}(\Sigma)}$ for some non-empty set $\Sigma \subseteq \Omega$. Hence $\mathcal{T}_\Omega$ has precisely $2^{|\Omega|} - 1$ maximal right congruences.

Left congruences of $\mathcal{T}_\Omega$

Theorem (C., Mitchell, and Péresse 2025+)

Let Ω be a finite set. Then $\mathcal{T}_\Omega$ has precisely $B(|\Omega|) - 1$ maximal left congruences.

The case when $|\Omega| = \infty$

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Let Ω be an infinite set. Then $\mathcal{T}_\Omega$ has precisely $2^{2^{|\Omega|}}$ maximal right congruences.

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Lemma

Let N be an LDC submonoid of $\mathcal{T}_\Omega$. Then the set $\{(\Omega)f : f \in N\}$ is a filter on $\mathcal{P}(\Omega)$.

Lemma

Let $\mathcal{F}$ be an ultrafilter on $\mathcal{P}(\Omega)$. Then the submonoid

$$\mathrm{End}(F) = \{\, f \in \mathcal{T}_\Omega \mid \forall \Sigma \in \mathcal{F},\ (\Sigma)f \in \mathcal{F} \,\}$$

is a maximal LDC submonoid of $\mathcal{T}_\Omega$.

Future work

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- What are the left congruences of $\mathcal{T}_\Omega$, when $|\Omega| = \infty$?

Future work

- Apply techniques to other transformation monoids, diagram monoids etc.
- Extend method to larger classes of one-sided congruences, e.g. meet irreducible one-sided congruences.
- What are the left congruences of $\mathcal{T}_{\Omega}$, when $|\Omega| = \infty$?
- Describe the lattice of right/left congruences modulo the lattice of LDC/RDC submonoids.

Thank you for your attention!

References I

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